Strengthening of Mixed Integer Linear Program Bounds using Variable Splitting

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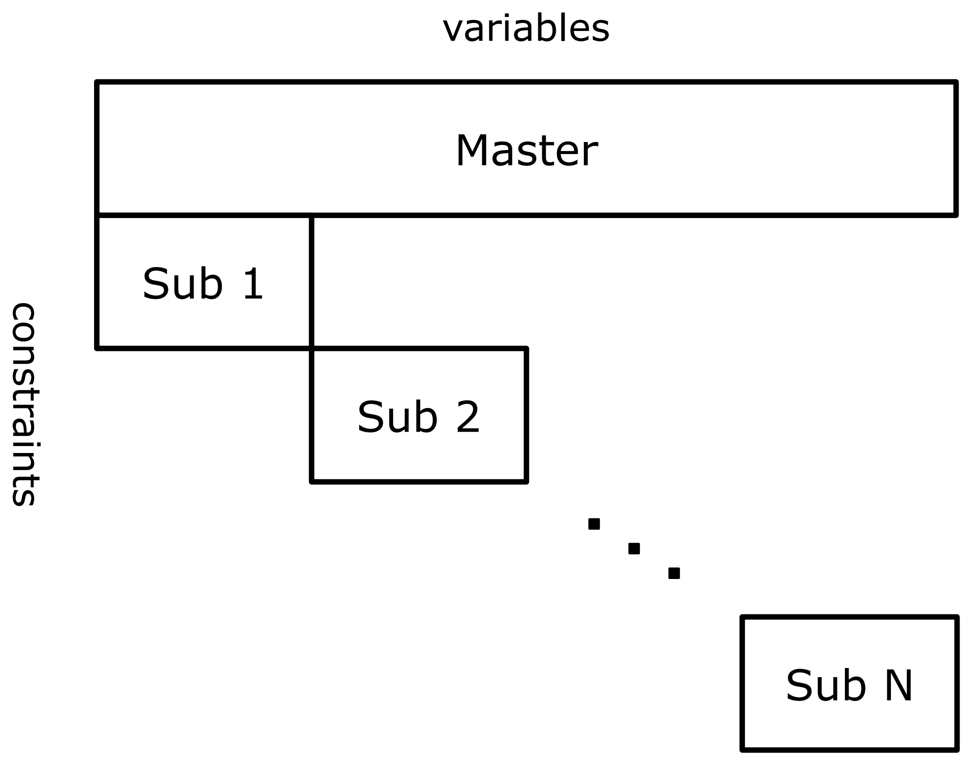
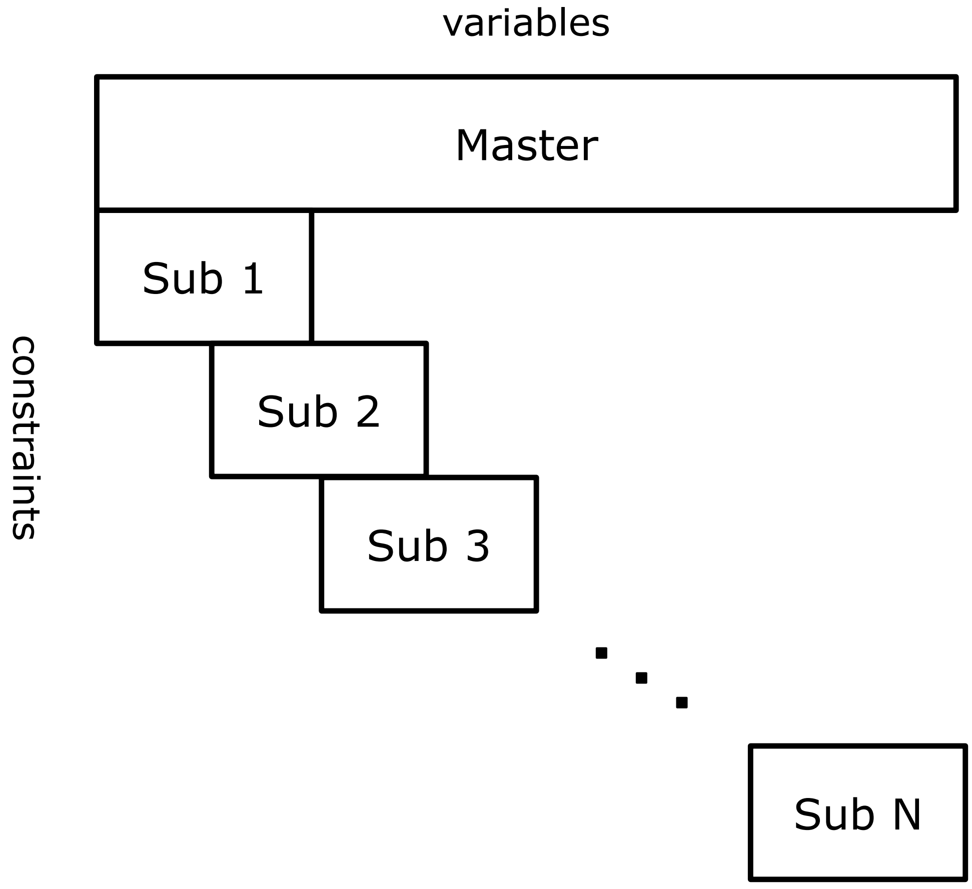
Dantzig Wolfe decomposition

- Splitup Mixed Integer Linear Program into master and sub problems
- Relax master, keep integrality in sub - Better LP bounds



Overlapping sub problems (Variable Splitting)

- Allow variables to be in more than one sub problem



Variable Splitting

$$\max Z = x_2$$

s.t.

$$-\frac{1}{2}x_1 + x_2 \geq 0$$

$$x_1 + \frac{10}{7}x_2 \leq 10$$

$$x_1 - \frac{3}{7}x_2 \leq 7$$

$$-\frac{7}{10}x_1 + x_2 \leq 0$$

$$x_1 + \frac{8}{6}x_2 \geq 8$$

$$x_1, x_2 \in \mathbb{Z}_+$$

Variable Splitting

$$\max Z = x_2$$

s.t.

$$-\frac{1}{2}x_1 + x_2 \geq 0$$

$$x_1 + \frac{10}{7}x_2 \leq 10$$

sub 1

$$x_1 - \frac{3}{7}x_2 \leq 7$$

$$-\frac{7}{10}x_1 + x_2 \leq 0$$

$$x_1 + \frac{8}{6}x_2 \geq 8$$

sub 2

$$x_1, x_2 \in \mathbb{Z}_+$$

Variable Splitting

$$\max Z = x_2$$

s.t.

$$-\frac{1}{2}x_1 + x_2 \geq 0$$

$$x_1 + \frac{10}{7}x_2 \leq 10$$

sub 1

$$x_1 - \frac{3}{7}x_2 \leq 7$$

$$-\frac{7}{10}x_1 + x_2 \leq 0$$

$$x_1 + \frac{8}{6}x_2 \geq 8$$

sub 2

$$x_1, x_2 \in \mathbb{Z}_+$$

$$\max Z = x_2$$

s.t.

$$x_1 = \bar{x}_1$$

$$x_2 = \bar{x}_2$$

master

Variable Splitting

$$\max Z = x_2$$

s.t.

$$-\frac{1}{2}x_1 + x_2 \geq 0$$

$$x_1 + \frac{10}{7}x_2 \leq 10$$

sub 1

$$x_1 - \frac{3}{7}x_2 \leq 7$$

$$-\frac{7}{10}x_1 + x_2 \leq 0$$

$$x_1 + \frac{8}{6}x_2 \geq 8$$

sub 2

$$x_1, x_2 \in \mathbb{Z}_+$$

$$\max Z = x_2$$

s.t.

$$x_1 = \bar{x}_1$$

$$x_2 = \bar{x}_2$$

master

$$-\frac{1}{2}x_1 + x_2 \geq 0$$

$$x_1 + \frac{10}{7}x_2 \leq 10$$

$$x_1, x_2 \in \mathbb{Z}_+$$

sub 1

Variable Splitting

$$\max Z = x_2$$

s.t.

$$-\frac{1}{2}x_1 + x_2 \geq 0$$
$$x_1 + \frac{10}{7}x_2 \leq 10$$

sub 1

$$x_1 - \frac{3}{7}x_2 \leq 7$$
$$-\frac{7}{10}x_1 + x_2 \leq 0$$
$$x_1 + \frac{8}{6}x_2 \geq 8$$

sub 2

$$x_1, x_2 \in \mathbb{Z}_+$$

$$\max Z = x_2$$

s.t.

$$x_1 = \bar{x}_1$$
$$x_2 = \bar{x}_2$$

master

$$-\frac{1}{2}x_1 + x_2 \geq 0$$
$$x_1 + \frac{10}{7}x_2 \leq 10$$
$$x_1, x_2 \in \mathbb{Z}_+$$

sub 1

$$\bar{x}_1 + \frac{10}{7}\bar{x}_2 \leq 7$$
$$-\frac{10}{7}\bar{x}_1 + \bar{x}_2 \leq 0$$
$$\bar{x}_1 + \frac{8}{6}\bar{x}_2 \geq 8$$
$$\bar{x}_1, \bar{x}_2 \in \mathbb{Z}_+$$

sub 2

Background

- Guignard, Monique, and Siwhan Kim. "Lagrangean decomposition: A model yielding stronger Lagrangean bounds." *Mathematical programming* 39, no. 2 (1987): 215-228.

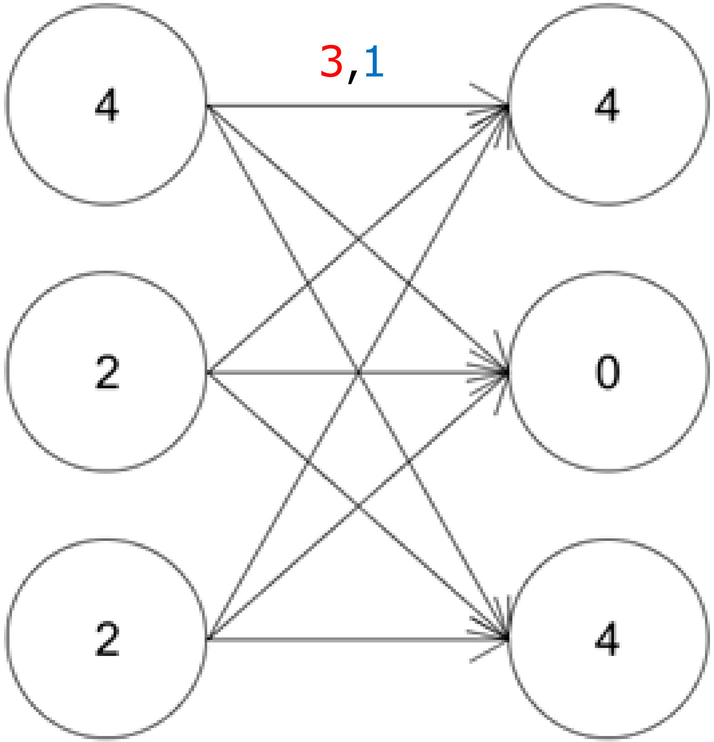
- Jörnsten, Kurt O., Mikael Näsberg, and Per A. Smeds. *Variable splitting: A new Lagrangean relaxation approach to some mathematical programming models*. Universitetet i Linköping/Tekniska Högskolan i Linköping. Department of Mathematics, 1985.

AutoDec

- Generic decomposition framework for MILPs (similar to GCG, BaPCod and DIP)
- Input: lpfile and desired decomposition
- Dantzig Wolfe decomposition and variable splitting
- Solve decomposition using column generation
- Fully implemented in Julia
- BoxPen stabilization

The Fixed Charge Transportation Problem (FCTP)

- Ship commodities from left to right
- Fixed cost of opening arc
- Variable cost per unit sent



FCTP model

$$\min \sum_{i\in S}\sum_{t\in T}(c_{ij}x_{ij} + f_{ij}y_{ij})$$

subject to

$$\sum_{i\in T} x_{ij} = a_i \qquad \forall i \in S \qquad (1)$$

$$\sum_{i\in S} x_{ij} = b_j \qquad \forall j \in T \qquad (2)$$

$$x_{ij} \leq m_{ij}y_{ij} \qquad \forall i \in S, j \in T \qquad (3)$$

Flow on arc →

$$x_{ij} \in \mathbb{Z}_+ \qquad \forall i \in S, j \in T \qquad (4)$$

Arc open or closed →

$$y_{ij} \in \{0,1\} \qquad \forall i \in S, j \in T \qquad (5)$$

FCTP model

$$\min \sum_{i \in S} \sum_{t \in T} (c_{ij} x_{ij} + f_{ij} y_{ij})$$

subject to

$$\sum_{i \in T} x_{ij} = a_i \qquad \forall i \in S \qquad (1)$$ ⟵ Sources

$$\sum_{i \in S} x_{ij} = b_j \qquad \forall j \in T \qquad (2)$$ ⟵ Sinks

$$x_{ij} \leq m_{ij} y_{ij} \qquad \forall i \in S, j \in T \qquad (3)$$

$$x_{ij} \in \mathbb{Z}_+ \qquad \forall i \in S, j \in T \qquad (4)$$

$$y_{ij} \in \{0, 1\} \qquad \forall i \in S, j \in T \qquad (5)$$

FCTP model

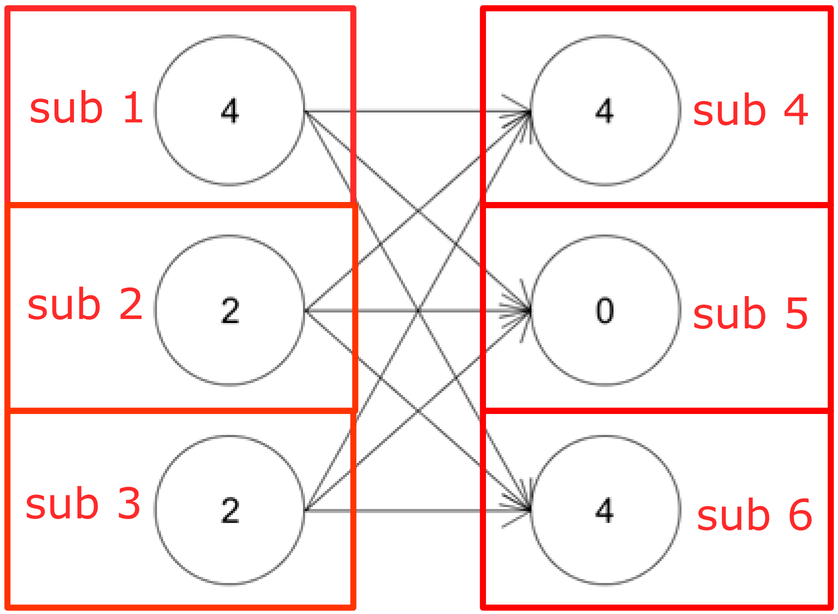
$$\min \sum_{i \in S} \sum_{t \in T} (c_{ij} x_{ij} + f_{ij} y_{ij})$$

subject to

$$\begin{aligned}
\sum_{i \in T} x_{ij} &= a_i & \forall i \in S \quad (1)\\
\sum_{i \in S} x_{ij} &= b_j & \forall j \in T \quad (2)\\
x_{ij} &\leq m_{ij} y_{ij} & \forall i \in S, j \in T \quad (3)\\
x_{ij} &\in \mathbb{Z}_+ & \forall i \in S, j \in T \quad (4)\\
y_{ij} &\in \{0, 1\} & \forall i \in S, j \in T \quad (5)
\end{aligned}$$

$$m_{ij} = \min\{a_i, b_j\}$$

The FCTP
Variable splitting



$$\min \sum_{i\in I, j\in J} \frac{1}{2}\left(c_{ij}x_{ij} + c_{ij}\bar{x}_{ij} + f_{ij}y_{ij} + f_{ij}\bar{y}_{ij}\right)$$

s.t.

$$x_{ij} = \bar{x}_{ij} \forall i \in I, j \in J$$

$$y_{ij} = \bar{y}_{ij} \forall i \in I, j \in J$$

master

$$\sum_{j\in J} x_{ij} = a_i \forall i \in I$$

sub

$$\sum_{i\in I} \bar{x}_{ij} = b_j \forall j \in J$$

$$x_{ij} \le m_{ij}y_{ij} \forall i \in I, j \in J$$

$$\bar{x}_{ij} \le m_{ij}\bar{y}_{ij} \forall i \in I, j \in J$$

$$x_{ij} \in \mathbb{Z}^+ \forall i \in I, j \in J, y_{ij} \in \{0,1\} \forall i \in I, j \in J$$

$$\bar{x}_{ij} \in \mathbb{Z}^+ \forall i \in I, j \in J, \bar{y}_{ij} \in \{0,1\} \forall i \in I, j \in J$$

Mingozzi, Aristide, and Roberto Roberti. "An Exact Algorithm for the Fixed Charge Transportation Problem Based on Matching Source and Sink Patterns." *Transportation Science* (2017).

The FCTP
Arc quantity equations

Substitute: $x_{ijq} = 1$ iff x_{ij} = q

$$\min \quad \sum_{i\in I, j\in J} \left(\sum_{q=1}^{m_{ij}} c_{ij} \cdot q \cdot x_{ijq} + f_{ij} \cdot y_{ij} \right)$$

s.t.

$$\sum_{q=1}^{m_{ij}} x_{ijq} \leq 1 \forall i \in I, j \in J$$

$$\sum_{j\in J} \sum_{q=1}^{m_{ij}} q \cdot x_{ijq} = a_i \forall i \in I$$

$$\sum_{i\in I} \sum_{q=1}^{m_{ij}} q \cdot x_{ijq} = b_j \forall j \in J$$

$$\sum_{q=1}^{m_{ij}} q \cdot x_{ijq} \leq m_{ij} \cdot y_{ij} \forall i \in I, j \in J$$

$$x_{ijq} \in \{0,1\} \forall i \in I, j \in J, q \in \{1, ..., m_{ij}\}, y_{ij} \in \{0,1\} \forall i \in I, j \in J$$

$x = \bar{x}$ constraint becomes: $x_{ijq} = \bar{x}_{ijq} \forall i \in I, j \in J, q \in \{1, ..., m_{ij}\}$

The FCTP
Results using variable splitting

size		MIP			LP	
sources	sinks	best UB	time	gap	lp	gap
15	10	47118	22.6	0.0%	41571.1	11.8%
15	10	52938	34.1	0.0%	46188.9	12.7%
15	10	58738	68.1	0.0%	50729.7	13.6%
10	15	43445	43.4	0.0%	38594.8	11.2%
10	15	48466	36.7	0.0%	42792.2	11.7%
10	15	53387	40.0	0.0%	46929.5	12.1%
20	20	57159	3619.3	2.6%	47798.0	16.4%
20	20	64630	3626.9	3.2%	53784.8	16.8%
20	20	72984	3603.1	5.5%	59688.8	18.2%

The FCTP
Results using variable splitting

size		MIP			LP		VarSplit		
sources	sinks	best UB	time	gap	lp	gap	relax	gap	time
15	10	47118	22.6	0.0%	41571.1	11.8%	44061.9	6.5%	19.2
15	10	52938	34.1	0.0%	46188.9	12.7%	49227.5	7.0%	18.7
15	10	58738	68.1	0.0%	50729.7	13.6%	54373.2	7.4%	18.9
10	15	43445	43.4	0.0%	38594.8	11.2%	40681.8	6.4%	18.5
10	15	48466	36.7	0.0%	42792.2	11.7%	45345.8	6.4%	18.3
10	15	53387	40.0	0.0%	46929.5	12.1%	49867.6	6.6%	18.7
20	20	57159	3619.3	2.6%	47798.0	16.4%	53487.2	6.4%	26.5
20	20	64630	3626.9	3.2%	53784.8	16.8%	60229.2	6.8%	27.8
20	20	72984	3603.1	5.5%	59688.8	18.2%	66924.8	8.3%	26.7

The FCTP
Results using variable splitting

size		MIP			LP		VarSplit			VarSplit + QuantEq		
sources	sinks	best UB	time	gap	lp	gap	relax	gap	time	relax	gap	time
15	10	47118	22.6	0.0%	41571.1	11.8%	44061.9	6.5%	19.2	47118.0	0.0%	97.5
15	10	52938	34.1	0.0%	46188.9	12.7%	49227.5	7.0%	18.7	52938.0	0.0%	107.8
15	10	58738	68.1	0.0%	50729.7	13.6%	54373.2	7.4%	18.9	58738.0	0.0%	123.6
10	15	43445	43.4	0.0%	38594.8	11.2%	40681.8	6.4%	18.5	43320.4	0.3%	115.5
10	15	48466	36.7	0.0%	42792.2	11.7%	45345.8	6.4%	18.3	48463.4	0.0%	105.1
10	15	53387	40.0	0.0%	46929.5	12.1%	49867.6	6.6%	18.7	53387.0	0.0%	80.8
20	20	57159	3619.3	2.6%	47798.0	16.4%	53487.2	6.4%	26.5	57159.0	0.0%	326.2
20	20	64630	3626.9	3.2%	53784.8	16.8%	60229.2	6.8%	27.8	64630.0	0.0%	380.7
20	20	72984	3603.1	5.5%	59688.8	18.2%	66924.8	8.3%	26.7	72070.0	1.3%	411.0

Single commodity, fixed charge, network flow problem (SCFCNFP)

- Fixed cost of opening arc
- Variable cost per unit sent
- Positive numbers in nodes: demand
- Negative numbers: supply

$$\min \sum_{a \in A} c_a x_a + \sum_{a \in A} f_a y_a$$

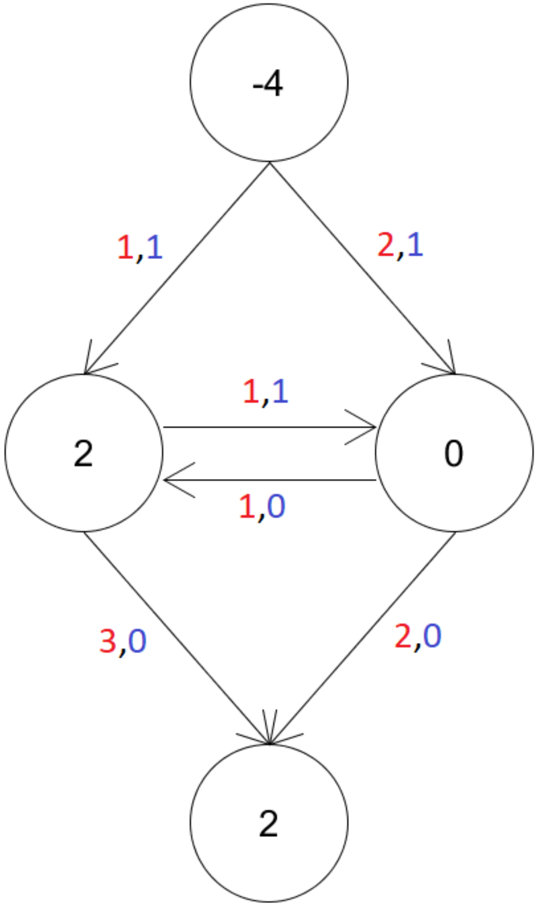
s.t.

$$\sum_{a \in \delta^-(i)} x_a - \sum_{a \in \delta^+(i)} x_a = b_i \qquad \forall i \in V$$

$$x_a \leq U y_a \qquad \forall a \in A$$

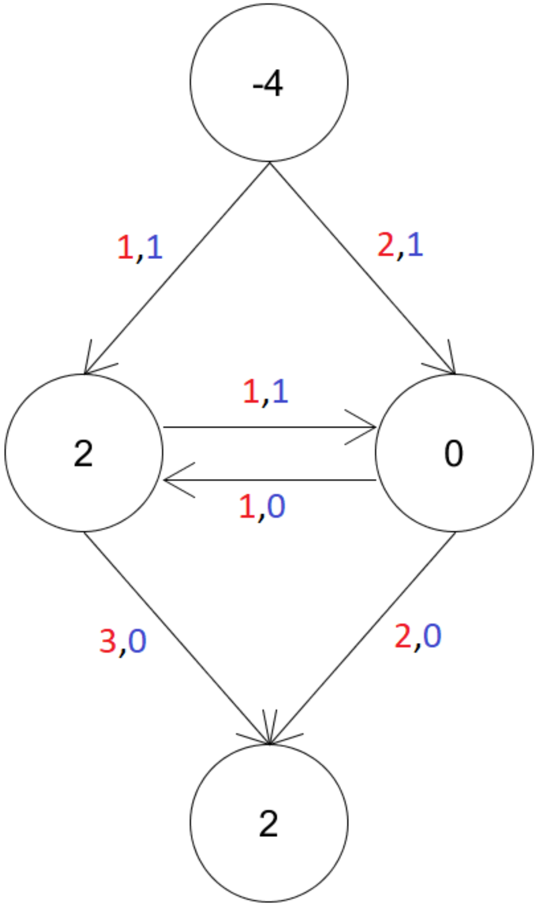
$$x_a \in \mathbb{Z}_+ \qquad \forall a \in A$$

$$y_a \in \{0, 1\} \qquad \forall a \in A$$



Single commodity, fixed charge, network flow problem (SCFCNFP)

- One sub-problem per node.
- x_a and y_a variables are split between sub-problems.



$$\min \sum_{a \in A} c_a x_a + \sum_{a \in A} f_a y_a$$

s.t.

$$\sum_{a \in \delta^-(i)} x_a - \sum_{a \in \delta^+(i)} x_a = b_i \qquad \forall i \in V$$

$$x_a \leq U y_a \qquad \forall a \in A$$

$$x_a \in \mathbb{Z}_+ \qquad \forall a \in A$$

$$y_a \in \{0, 1\} \qquad \forall a \in A$$

The SCFCNFP
Sup-problem cuts

Sub cuts:

$$y_{ij} \leq x_{ij} \forall (i,j) \in A | f_{ij} \geq 0$$

$$\sum_{j \in V} y_{ij} \leq U \sum_{j \in V} y_{ji} \; \forall i \in V | d_i = 0$$

$$\sum_{j \in V} y_{ji} \leq U \sum_{j \in V} y_{ij} \; \forall i \in V | d_i = 0$$

$$y_{ij} + y_{ji} \leq 1 \forall (i,j) \in A | (j,i) \in A$$

The SCFCNFP DiCuts

Master cuts:

$$\sum_{(i,j)\in A|i\in\mathscr{V},j\in V/\mathscr{V}} y_{ij} \geq 1 \forall \mathscr{V} \subset V | \sum_{i\in\mathscr{V}} b_i < 0$$

$$\sum_{(i,j)\in A|i\in V/\mathscr{V},j\in\mathscr{V}} y_{ij} \geq 1 \forall \mathscr{V} \subset V | \sum_{i\in\mathscr{V}} b_i > 0$$

Ortega, Francisco, and Laurence A. Wolsey. "A branch-and-cut algorithm for the single-commodity, uncapacitated, fixed-charge network flow problem." *Networks: An International Journal* 41.3 (2003): 143-158.

The SCFCNFP
Results using variable splitting

size		MIP			LP	
vertices	arcs	best UB	gap	time	relax	gap
500	1250	85.0	0.0%	0.8	33.3	60.9%
500	1250	144.0	0.0%	2.6	41.1	71.5%
500	1250	754.0	0.0%	14.4	40.4	94.6%
52	2652	1044.0	0.0%	5.8	130.1	87.5%
58	3306	13655.0	0.0%	9.6	2208.7	83.8%
200	740	43128.0	6.2%	3605.4	7911.5	81.7%
200	740	131407.0	0.6%	3606.4	86698.8	34.0%
200	740	30086.0	2.7%	3604.2	2292.5	92.4%
200	1188	55944.0	1.5%	3602.5	7860.8	85.9%
500	2988	175152.0	1.3%	3605.2	61188.7	65.1%
80	800	5332.0	0.6%	3602.7	3651.5	31.5%

Fixed Cost Network Flow Library (FCNetLib)
http://www.gamsworld.org/performance/fcnetlib/

The SCFCNFP
Results using variable splitting

size		MIP			LP		VarSplit		
vertices	arcs	best UB	gap	time	relax	gap	best LB	gap	time
500	1250	85.0	0.0%	0.8	33.3	60.9%	44.8	47.4%	23.8
500	1250	144.0	0.0%	2.6	41.1	71.5%	69.3	51.8%	39.4
500	1250	754.0	0.0%	14.4	40.4	94.6%	293.4	61.1%	50.9
52	2652	1044.0	0.0%	5.8	130.1	87.5%	650.5	37.7%	41.5
58	3306	13655.0	0.0%	9.6	2208.7	83.8%	9808.0	28.2%	187.9
200	740	43128.0	6.2%	3605.4	7911.5	81.7%	31253.5	27.5%	55.1
200	740	131407.0	0.6%	3606.4	86698.8	34.0%	118547.7	9.8%	57.4
200	740	30086.0	2.7%	3604.2	2292.5	92.4%	21840.2	27.4%	52.9
200	1188	55944.0	1.5%	3602.5	7860.8	85.9%	38585.5	31.0%	50.0
500	2988	175152.0	1.3%	3605.2	61188.7	65.1%	136838.5	21.9%	148.4
80	800	5332.0	0.6%	3602.7	3651.5	31.5%	4912.1	7.9%	33.9

The SCFCNFP
Results using variable splitting

size		MIP			LP		VarSplit			VarSplit + SubCuts + DiCuts		
vertices	arcs	best UB	gap	time	relax	gap	best LB	gap	time	best LB	gap	time
500	1250	85.0	0.0%	0.8	33.3	60.9%	44.8	47.4%	23.8	85.0	0.0%	42.5
500	1250	144.0	0.0%	2.6	41.1	71.5%	69.3	51.8%	39.4	144.0	0.0%	62.5
500	1250	754.0	0.0%	14.4	40.4	94.6%	293.4	61.1%	50.9	754.0	0.0%	101.2
52	2652	1044.0	0.0%	5.8	130.1	87.5%	650.5	37.7%	41.5	1044.0	0.0%	699.9
58	3306	13655.0	0.0%	9.6	2208.7	83.8%	9808.0	28.2%	187.9	13655.0	0.0%	3131.2
200	740	43128.0	6.2%	3605.4	7911.5	81.7%	31253.5	27.5%	55.1	38285.6	11.2%	948.0
200	740	131407.0	0.6%	3606.4	86698.8	34.0%	118547.7	9.8%	57.4	125237.2	4.7%	762.9
200	740	30086.0	2.7%	3604.2	2292.5	92.4%	21840.2	27.4%	52.9	27769.6	7.7%	1060.5
200	1188	55944.0	1.5%	3602.5	7860.8	85.9%	38585.5	31.0%	50.0	51624.8	7.7%	1376.5
500	2988	175152.0	1.3%	3605.2	61188.7	65.1%	136838.5	21.9%	148.4	163724.3	6.5%	3608.1
80	800	5332.0	0.6%	3602.7	3651.5	31.5%	4912.1	7.9%	33.9	5101.8	4.3%	210.8

The SCFCNFP
Results using variable splitting

size		MIP			LP		VarSplit			VarSplit + SubCuts + DiCuts			VarSplit + all cuts		
vertices	arcs	best UB	gap	time	relax	gap	best LB	gap	time	best LB	gap	time	best LB	gap	time
500	1250	85.0	0.0%	0.8	33.3	60.9%	44.8	47.4%	23.8	85.0	0.0%	42.5	85.0	0.0%	56.2
500	1250	144.0	0.0%	2.6	41.1	71.5%	69.3	51.8%	39.4	144.0	0.0%	62.5	144.0	0.0%	81.7
500	1250	754.0	0.0%	14.4	40.4	94.6%	293.4	61.1%	50.9	754.0	0.0%	101.2	754.0	0.0%	145.5
52	2652	1044.0	0.0%	5.8	130.1	87.5%	650.5	37.7%	41.5	1044.0	0.0%	699.9	1044.0	0.0%	1013.0
58	3306	13655.0	0.0%	9.6	2208.7	83.8%	9808.0	28.2%	187.9	13655.0	0.0%	3131.2	13650.2	0.0%	3620.0
200	740	43128.0	6.2%	3605.4	7911.5	81.7%	31253.5	27.5%	55.1	38285.6	11.2%	948.0	39223.5	9.1%	3613.0
200	740	131407.0	0.6%	3606.4	86698.8	34.0%	118547.7	9.8%	57.4	125237.2	4.7%	762.9	125263.1	4.7%	3760.6
200	740	30086.0	2.7%	3604.2	2292.5	92.4%	21840.2	27.4%	52.9	27769.6	7.7%	1060.5	28709.0	4.6%	3613.0
200	1188	55944.0	1.5%	3602.5	7860.8	85.9%	38585.5	31.0%	50.0	51624.8	7.7%	1376.5	52805.4	5.6%	3607.5
500	2988	175152.0	1.3%	3605.2	61188.7	65.1%	136838.5	21.9%	148.4	163724.3	6.5%	3608.1	163724.3	6.5%	3608.4
80	800	5332.0	0.6%	3602.7	3651.5	31.5%	4912.1	7.9%	33.9	5101.8	4.3%	210.8	5301.3	0.6%	3607.3

The SCFCNFP
Results using variable splitting

size		MIP				VarSplit + all cuts				
vartices	arcs	best known	UB at end	gap	solvetime	best LB	gap	time	time pricing	time LP
200	740	43128	43655	6.7%	43242	41757.3	3.2%	43216	37907	4654
200	740	131407	131407	0.6%	43247	127737.4	2.8%	43257	42818	251
200	740	30086	30200	3.7%	43200	29299.3	2.6%	20319	12800	6450
200	1188	55944	55944	1.1%	43229	55192.2	1.3%	43208	39850	2833
500	2988	175050	175050	1.3%	43231	162270.7	7.3%	43320	42556	424
80	800	5332	5332	0.0%	27395	5318.0	0.3%	6089	5814	192

The SCFCNFP
Results using variable splitting

size		MIP				VarSplit + all cuts				
vartices	arcs	best known	UB at end	gap	solvetime	best LB	gap	time	time pricing	time LP
200	740	43128	43655	6.7%	43242	41757.3	3.2%	43216	37907	4654
200	740	131407	131407	0.6%	43247	127737.4	2.8%	43257	42818	251
200	740	30086	30200	3.7%	43200	29299.3	2.6%	20319	12800	6450
200	1188	55944	55944	1.1%	43229	55192.2	1.3%	43208	39850	2833
500	2988	175050	175050	1.3%	43231	162270.7	7.3%	43320	42556	424
80	800	5332	5332	0.0%	27395	5318.0	0.3%	6089	5814	192

Use heuristic to solve the sub problems.

Conclusion

- AutoDec: Generic framework for doing Dantzig-Wolfe decomposition and variable splitting.
- Quantity constraints by Mingozzi and Roberti can be generalized to many problem classes.
- Quantity constraints helps a lot on both problems, but are slow in current implementation.
- Goal: add branch-and-price + Let the program decide the decomposition itself.